Constrained three parameter AVO inversion and uncertainty analysis

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Summary

Bayes' theorem is used to derive a 3 parameter non-linear AVO inversion. Geologic constraints based on available well control or rock physical relationships are incorporated to help stabilize the solution. Parameter uncertainty estimates arise naturally as part of the derivation and provide estimates of the reliability of the different parameters. The resulting parameter and uncertainty estimates may be transformed to a variety of elastic and rock physical AVO attributes popular in the literature using a transform matrix.

Introduction

The elastic parameters may be estimated, using a linearized approximation of the Zoeppritz equation such as Aki and Richards (1980)

$$r(\theta)=\frac{1}{2}\left(1-4\gamma^2\sin^2\theta\right)\frac{\Delta\rho}{\rho}+\frac{1}{2}\frac{\Delta\alpha}{\alpha}\frac{1}{\cos^2\theta}-4\gamma^2\left(\frac{\Delta\beta}{\beta}\right)\sin^2\theta, \qquad (1)$$

where r is the angle dependent reflectivity. The parameters α, β, ρ, • respectively are the average p-wave velocity, s-wave velocity, density and the ratio of S-velocity to P-velocity across the interface. The variable θ is the average angle of incidence and $\Delta\alpha$, $\Delta\beta$, $\Delta\rho$ are the change in p-wave velocity, s-wave velocity and density. Equation 1 may be written in matrix form Gm=d where **G** is the linear operator, **m** the unknown parameter vector containing the velocity and density reflectivity $[\Delta\alpha/\alpha, \Delta\beta/\beta, \Delta\rho/\rho]^t$ and **d** the input data vector (offset dependent reflectivity).

In practice, equation (1) is rarely inverted for. For conventional acquisition geometries and noise levels, equation (1) is ill-conditioned. That is, a small amount of noise will result in large parameter deviations. This problem becomes worse as the range of angles used in the inversion becomes smaller. Various authors (Shuey (1985), Smith and Gidlow (1987), and Fatti et al (1994), among others) rearrange equation (1) to solve for other parameterizations. In implementing these schemes, hard constraints are usually implemented either explicitly or implicitly to improve the stability of the problem. Smith and Gidlow (1987) use the Gardner equation (Gardner et al., 1973) to remove the density reflectivity thus improving the stability of the problem. The Shuey and Fatti equations are generally both solved using only the first two terms implicitly constraining the 3rd term's reflectivity to zero.

Rather than using hard constraints, this paper uses soft constraints from the well control or from rock physical relationships. The degree to which the constraint influences the solution is dependent on the signal to noise level of the data and the acquisition geometry. In the case of good signal-to-noise data and a large angle range, the constraints only influence the solution in a minor way. Under these conditions the density reflectivity might be reliably solved for. For poor signal to noise ratio data or a data with limited angle range the constraints will dominate the solution. Parameter estimates will have greater uncertainty and quality control displays must be relied upon to determine if the estimate is useable (Downton et al, 2000)

Bayes' theorem provides a convenient theoretical framework to do this. This paper first reviews Bayes' theorem, the Likelihood function, and the prior constraints. It is shown how well control or rock physical relationships can be used to construct a prior probability function. Then, by combining the Likelihood function and the a prior probability function, a non-linear inversion algorithm is derived. Next, the reliability of the estimates is discussed. It is shown that the parameter estimate and uncertainty can be transformed to other AVO attributes that might be more suitable to interpret the data. Lastly, the algorithm is demonstrated on synthetic seismic data.

Theory

Bayesian Inversion

Bayes' theorem provides a theoretical framework to make probabilistic estimates of the unknown parameters **m** from uncertain data and a priori information. The resulting probabilistic parameter estimates are called the Posterior Probability Distribution function (PDF). The PDF written as P(**m**|**d**,I) symbolically indicates the probability of the parameter vector **m** given the data vector **d** (offset dependent reflectivity) and information I. Bayes' theorem

$$P(\mathbf{m}\,|\,\mathbf{d},I)=\frac{P(\mathbf{d}\,|\,\mathbf{m},I)P(\mathbf{m}\,|\,I)}{P(\mathbf{d}\,|\,I)}, \qquad (2)$$

calculates the PDF from the likelihood function P(**d**|**m**,I) and a priori probability function P(**m**|I). The denominator P(**d**|I) is a normalization function which may be ignored if only the shape of the PDF is of interest

$$P(\mathbf{m}\,|\,\mathbf{d},I)=P(\mathbf{d}\,|\,\mathbf{m},I)P(\mathbf{m}\,|\,I). \qquad (3)$$

The most likely estimate occurs at the maximum of the PDF. The uncertainty of the parameter estimate is proportional to the width of the PDF.

Likelihood function
If we assume uniform uncorrelated Gaussian noise then the likelihood function may be written as (Sivia, 1996)

$$P(\mathbf{d}\mid\mathbf{m},I)\propto\sigma^{-N}\exp\left[-\frac{\sum_{k=1}^{N}\left(\sum_{i=1}^{M}G_{ki}m_i-d_k\right)^2}{2\sigma^2}\right], \qquad (4)$$

where σ^2 is the variance of the noise. In the case of uniform priors, Bayesian inversion is equivalent to maximum likelihood inversion.

For AVO inversion, because of the small number of parameters solved for, it is possible to visualize the PDF. If the parameter vector **m** had only one element, the PDF would be a Gaussian function. If the parameter vector **m** had two elements, the PDF would be a bivariate Gaussian function and an equiprobable solution would be an ellipse. For the case of AVO inversion, where there are 3 parameters, the PDF is a multivariate Gaussian function where an equiprobable solution is an ellipsoid. Typically the ellipsoid is quite elongated along the density reflectivity axis. The solutions are non-physical when the reflectivity is greater than 1.

A priori constraints
One way to reduce the uncertainty is to impose constraints on the solution. For example, non-physical solutions can be excluded from the solution space. This can be written in terms of a probability distribution where physical solutions are equiprobable and non-physical solutions given zero probability.

It is not necessarily desirable to assign uniform probabilities over the range of physically valid reflectivity. The stratigraphic sequence is a result of cyclic geologic processes that result in reflectivity probability functions, which may be reasonably described by common probability functions. The normal distribution was found to reasonably describe the statistics of logs for this work in Western Canada. Other probability functions may be used if appropriate, but for this paper the normal probability function is used. The joint probability distribution for the P-velocity, S-velocity and density reflectivity is the multi-variate Gaussian distribution that is parameterized by a covariance matrix. The diagonal elements of the covariance matrix are the variances of the P-velocity, S-velocity and density reflectivity. The off-diagonal elements describe how correlated the P-velocity, S-velocity and density reflectivity are.

From rock physics studies it has been empirically observed that the P-velocity, S-velocity and density are correlated. The mudrock relationship (Castagna et al, 1985) provides a relationship linking P-velocity and S-velocity reflectivity, $R_{Vs}=mR_{Vp}$. The Gardner relationship (Gardner et al, 1973) provides a relationship between the P-velocity and density reflectivity, $R_d=gR_{Vp}$. Potter and Stewart (1998) observed a similar relationship between S-velocity and density reflectivity, $R_d=fR_{Vs}$. These parameters and their correlation coefficients r_1, r_2 and r_3 can be calculated from the local well control. From this the parameter covariance matrix $\mathbf{C}_m$, which defines the multi-variate Gaussian distribution can be constructed.

$$\begin{bmatrix}\sigma^2_{R_{Vp}} & \sigma_{R_{Vp}R_{Vs}} & \sigma_{R_{Vp}R_{den}}\\ \sigma_{R_{Vp}R_{Vs}} & \sigma^2_{R_{Vs}} & \sigma_{R_{Vs}R_{den}}\\ \sigma_{R_{Vp}R_{den}} & \sigma_{R_{Vs}R_{den}} & \sigma^2_{R_{den}}\end{bmatrix}=\sigma^2_{R_{Vp}}\begin{bmatrix}1 & \mathrm{m} & \mathrm{g}\\ \mathrm{m} & \frac{\mathrm{m}}{\mathrm{r}_1^{\,2}} & \mathrm{f}\\ \mathrm{g} & \mathrm{f} & \frac{\mathrm{g}}{\mathrm{r}^2{}_2}\end{bmatrix}. \qquad (5)$$

In practice it is more efficient to calculate this directly from the sample statistics, but the proceeding analysis provides physical significance to each of the terms in the parameter covariance matrix. In addition, in areas with limited well control or missing information, published values may be used to help construct the covariance matrix.

The resulting a priori probability function is the multi-variate Gaussian probability function

$$P(\mathbf{m}\mid I)\propto\exp\left[-\frac{1}{2}\mathbf{m}^T\frac{\mathbf{C}_m^{-1}}{\eta^2}\mathbf{m}\right], \qquad (6)$$

where η a global scale factor to account for the arbitrary scaling of the seismic data.

Nonlinear inversion
The Likelihood function (equation 4) may be combined with the a priori probability function (equation 6) using Bayes' Theorem equation (3). Since there is no explicit interest in the variance σ or the scalar η, both are marginalized (Sivia, 1996). The most likely solution can then be found by finding where the probability function is stationary. This involves taking the partial derivatives with respect to each parameter, setting the result to zero and solving the set of simultaneous equations. This results in the nonlinear equation

$$\mathbf{m}=\left[\mathbf{G}^T\mathbf{G}+\frac{2\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}}{(N-1)Q}\mathbf{C}^{-1}{}_m\right]^{-1}\mathbf{G}^T\mathbf{d}, \qquad (7)$$

where ε=Gm-d and $Q=\mathbf{m}^T\mathbf{C}_m^{-1}\mathbf{m}$. The equation is weakly nonlinear and can be solved in an iterative fashion using Newton-Raphson. The term $\boldsymbol{\varepsilon}^{\mathbf{T}}\boldsymbol{\varepsilon}$ is an estimate of the RMS energy of the noise and Q the RMS energy of the signal. The ratio is therefore an estimate of the N/S ratio. The ratio acts as a weighting factor determining how much the prior constraints influence the solution. If the S/N is large, then the weighting factor is small and the constraints add little to the solution and vice versa.

Uncertainty analysis
The uncertainty of the parameter estimate is related to the width of the distribution. This can be calculated from the 2nd derivative evaluated at the parameter estimate. With the assumption of uniform uncorrelated Gaussian noise the uncertainty is described by the covariance matrix

$$\mathbf{C}_d=\left[\mathbf{G}^T\mathbf{G}+\frac{2\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}}{(N-1)Q}\mathbf{C}^{-1}{}_m\right]^{-1}. \qquad (8)$$

The diagonal of the covariance matrix represents the variance of each parameter estimate. The off-diagonal element represents the degree of correlation between the errors. (Downton et al., 2000)

It is also important to understand how much the constraints are influencing the solution. The uncertainty can also be calculated if the constraints were not included. The ratio of these two uncertainty estimates give a sense for how much of the solution is coming from the data and how much from prior knowledge. To make accurate predictions about the subsurface the parameter estimate of interest should be largely coming from the data.

Transform matrix

In this paper the P-velocity, S-velocity and density reflectivity are estimated. In the literature, equation (1) has be rearranged to solve for impedance reflectivity (Fatti et al, 1994), Lame reflectivity (Gray et al, 1999), geometric parameters A,B,C (Shuey, 1985) and many others. This being the case, it is simple to construct a transform matrix to transform from velocity reflectivity to any of these other attributes $\mathbf{m}' = \mathbf{T}\mathbf{m}$ where T is the transform matrix and $\mathbf{m}'$ is the new parameter set. The parameter uncertainty covariance matrix can also be transformed by $\mathbf{C}_m' = \mathbf{T}\mathbf{C}_m\mathbf{T}^T$. In this way different AVO attributes can be examined to see how they show off some particular geologic feature or anomaly. An attribute can be selected which highlights the objective the best. Of equal importance, the reliability of each of these attributes can also be examined to understand whether the anomaly is reliable or an artifact due to the noise.

Example

The method has been tested on both synthetic and real data. The synthetic data was generated based on two wells from Western Canada. Synthetic gathers were generated with a variety of different acquisition geometries to understand how the inversion would react to changes in fold, angle range and signal to noise. The constraints were constructed based on a composite of logs in each area. The results of the constrained inversion were transformed to impedance reflectivity and compared to the results of the 2 term Fatti equation. The results of the constrained S-impedance inversion were superior or equal to that of the 2 term Fatti equation. For high signal to noise levels (greater than 8 to 1) and large angle ranges (greater than 30 degrees) the inversion was able to predict the density reflectivity for some markers (Figure 1). These markers were significant in that the P-velocity and S-velocity were uncorrelated with the density so the prediction is coming from the data.

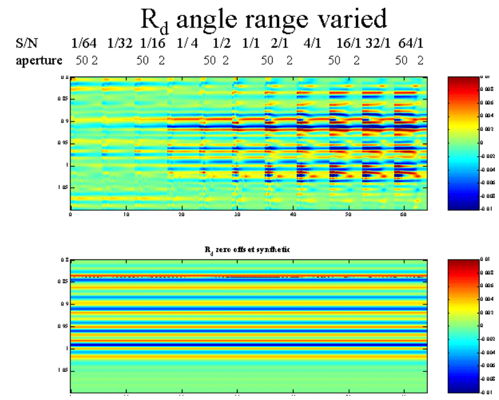

Figure 1: Comparison of density estimate from 3 term AVO inversion (top) and ideal synthetic (bottom) for AVO inversions on data with different angle ranges and S/N ratios. Note for high S/N ratios and large angle ranges it is possible to estimate the density reliably.

Conclusions

We have demonstrated a 3 parameter AVO inversion using soft constraints. The degree to which the constraints influence the solution is a function of the signal-to-noise ratio of the data. The constraints preferably should be calculated from local well control. If local well control is not available, values from the literature may be used. Velocity and density reflectivity are solved for, but can be transformed subsequently to virtually any other AVO attribute. Parameter uncertainty estimates are provided as part of the derivation and should be examined to determine the significance and reliability of a particular AVO attribute. This is particularly true of the density reflectivity. Density reflectivity may be reliably estimated for data with little noise and large angle range.

The results of the 3 term constrained AVO inversion are equivalent to the Smith and Gidlow AVO inversion if the a priori constraints define the density reflectivity as a linear function of the P-velocity reflectivity. Similarly, the results of the inversion are equivalent to the two term Fatti equation (Fatti et al, 1994) if the a priori information specifies the density reflectivity is zero. Lastly the three term AVO inversion is equivalent to the two term Shuey equation with the a priori constraint that the velocity reflectivity is zero. By choosing constraints based on local well control, honoring known rock physical relationships, and weighting the constraints based on the needs of the data, the results of the constrained 3 parameter AVO inversion should be more accurate than the aforementioned methods.

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