

## Anisotropy Kinematics Clarified through *Regime Rays* Model

William J. Vetter\*  
Kitchener, ON, Canada  
vetterb@sympatico.ca

### Summary

“So-called *rays*” in the anisotropy context are just to-line-loci-idealized energy transport channels (c.f. Cerveny 2001 p.28-30, p.103). They are (1) not *traveltimes* minimizing Fermat’s principle loci, (2) not pointwise Snell’s law conforming, (3) not direction-wise orthogonal to front tangent planes; (4) and, *velocity* or *slowness* magnitudes along loci directions differ from those in front-normal directions. All the understood attributes/characteristics of traditional *rays*, call those here “*proper rays*”, have been compromised through the inappropriate “so-called *rays*” labeling (likely from use initially in Rudzki’s lesser known 1913 paper).

I show here that “*proper rays*” in the conventional ray theory sense are involved in wavefront progression when triggered by idealized impulse excitation at a point source in an anisotropic medium segment. For homogeneous anisotropies those “*proper rays*” have principal segmentals oriented along front-normal directions with mode-specific front-normal velocity magnitudes that come, for given medium parameters and front-normal directions, from eigenvalue solutions of the Christoffel matrix; those principal segmentals are repetetively linked with two or single other-directed segmentals which manifest with physics-wise still puzzling velocities, forming thereby “*proper ray*” loci within the energy transport channels. Reordered representations, call them *regime rays*, clarify paths and along-paths velocity detail, they quantify “*proper rays*” *pathlengths* and *apparent-* (so-called *ray-*), *time-average-*, *path-mean-*, and *RMS- velocities*. All this stems from fine-structured medium heterogeneity not captured in present anisotropy kinematics models.

Anisotropy *regime rays* models explain manifesting kinematics credibly, and they can contribute important detail and capability to seismic velocity field modeling and seismic data inversing.

### Issues and Insights

Anisotropy has during recent decades become a mainstream theme and concern in seismic exploration (e.g. MacBeth and Lynn ed. SEG reprints 2000; broad literature). This theme prominence notwithstanding, formulations and models for energy transport loci and wavefront propagation in anisotropic medium segments in terms of ray theory (Rudzki 1911, 1913; Gassmann 1964, Chapman and Pratt 1992, Helbig 1994, Cerveny 2001, many others) convey a sense that something significant may have been overlooked (c.f. Vetter 1993, 1999, 2004; that prior work is here refined, elaborated, and extended).

In the pre-anisotropy era the word ray(s) was vocabulary and concept that linked to basic ray theory with Snell’s law [1621] and Fermat’s principle consistent loci [1657] (e.g. Aki and Richards

1980, ... ). That traditional meaning of ray(s) has now, in anisotropic medium propagation context, been categorically compromised through the substitution sense of “ray(s)” for what may most fittingly be deemed “to-line-loci-idealized *energy transport channels*” (c.f. Cerveny 2001, others). This stems likely from Rudzki’s 1913 paper (Slawinski 2002 translation from French) in which he requested for clarity on *the rays* within anisotropic medium segments, and on Snell’s-law-like constraints at transition boundaries between anisotropic segments. In effect Rudzki started his exploratory development with the for his time reasonable (but invalid) premise that the evolving smooth loci from shot to front points in context of wavefront progression were “*the rays*” ; thus scalar and now also vectored “ray-velocities” designations, as oriented lengths of “the rays” divided by associated *traveltimes*. Those are just *apparent velocities* associated with the “so-called *rays*”.

Towards clarifying the issues we need to recall some fundamentals: (1) Green’s equation re at-coordinate- frame-origin triggered disturbance in an isotropic or anisotropic medium,  $t v_N(\mathbf{n}; c_{ij}, \rho) = \mathbf{r}_E \cdot \mathbf{n}$ , eqn (1a) in the Appendix, (Love 1882/..1927, 1944 ch.8, Rudzki 1911); this equation models kinematics of ideal impulse progression and, in dual role for a fixed *traveltime* value, it encapsulates wavefront detail as the envelope of tangent planes at all front points  $\mathbf{r}_E$ .  $\mathbf{n}$  denotes front-normal directions and  $v_N$  is front-normal velocity at  $\mathbf{r}_E$  front-points; (2) elasticity theory fundamentals in Kelvin-Christoffel matrix encapsulation  $\Gamma(\mathbf{n}; a_{ij}, \rho)$  yielding, through elasticity mode-specific moduli  $M = \rho v_N^2 = \text{fn}(\Gamma(\mathbf{n}; a_{ij}, \rho))$ , the mode-specific front-normal velocities  $v_N(\mathbf{n}; c_{ij}, \rho)$  in Green’s equation; (3) Fermat’s principle (FP), adapted for seismic propagation as pointwise Snell’s law conforming *ray* loci, including direction discontinuities. This principle asserts that “proper rays” progress along pathtimes (traveltimes) minimizing loci.

FP is generally invoked in formulations as ‘extremals of Fermat’s functional’ (Rudzki 1913, ..., Chapman/ Pratt 1992, Cerveny 2001,...). Those formulations need categorically valid nominal “*proper rays*” loci. In anisotropy context such formulations are not credible. Why?, because the hypothesised nominal paths do not have adequate analytical/ parametric flexibility to represent the complexities of “*proper rays*”. This is hindsight from constraints that evolve from minimized traveltime in Green’s equation, for source and front points [ $\mathbf{r}_O \mathbf{r}_E$ ] deemed fixed, and front-normal direction  $\mathbf{n} = [n_x \ n_y \ n_z]$ , or equivalently spherical coordinate angles( $\theta, \phi$ ), deemed the parameters to be optimally assigned. The significant intermediate steps and final results from minimized traveltimes are detailed in the Appendix. First stage traveltime minimized constraints (2) and (3), combined with Green’s eqn (1b), give constraint eqn (4). In eqn (5) inversed from (4)  $\mathbf{v}_E$  depends on front-normal  $v_N$  and its directional derivatives re ( $\theta, \phi$ ). Eqn (7) shows  $\mathbf{v}_E$  dependence on  $v_N * \mathbf{n}$ , plus complicated weighting of the directional derivatives through matrix  $N = \text{fn}(n_x \ n_y \ n_z)$ . Note that eqn (7) detail differs from Helbig’s corresponding expression (1994, p.13 (1A.6) ) but here without ‘common dispersion’ term. Helbig’s likely not correct expression has been variously quoted/used elsewhere (e.g. Mensch/ Rasolofosaon 1997).

### **Regime Rays**

Relationships (8) [with two Pythagoras theorem encapsulations] and (9) [with orthogonally vectored detail] are variant forms of eqn (5) detail. The geometric representations/ visualizations of those constraints suggest strongly that “*proper rays*” [with representations as *regime rays*] are involved in the anisotropy kinematics.

We are conditioned to think of *rays* as kinematics representations valid essentially for high frequencies/short wavelengths re scale of medium segments/ heterogeneity. Still, fine scale propagation detail might be here involved during disturbance progression through a fine structured medium of adequate extent. “*Proper rays*” must be continuous, and detail/ artifacts/ transitions

encountered, like layers, layer boundary characteristics, cracks, voids with fluid inclusions, all deemed fine structured heterogeneity, ... , how any-which-way those would be optimally encapsulated through parameters of elasticity models, the pointwise velocity and direction changes must be then Snell's law conforming. That is feasible in ray models through linked front-normal directed segmentals together with some other-directed segmentals, in patterns likely captured in the *FP* constrained eqns (5, 7, 8, 9), principally (8). But for *regime rays* encapsulation, lengths detail of segmentals, while presumably linked to medium microstructure, is not important in context of spectral content of practical perturbations. We can thus idealize the segmentals as proportionally sized infinitessimals.

Conceptual re-ordering of infinitessimals encapsulates/ reveals/ clarifies the significant kinematics detail. *Regime rays* have just three or fewer components (Appendix Task-2 & eqns (10-16) ), each component the sum of same-direction infinitesimals. From those representations we discern then *pathlengths* (typically significantly larger than the apparent lengths), *pathtimes* (or *traveltimes*), and along paths velocity distributions with the significant characterizing means. It is through the great flexibility from linked segmentals in models that we can comprehend and analytically encapsulate the nuances in anisotropy kinematics, the association details for wavefronts and "so-called *rays*" progression velocities with their analytical front-normal representation counterparts, the cusping and multi-sheeting of qSV mode fronts, and more. The insights here portend potential for clarifying also puzzling polarization and dynamics issues in the anisotropy context.

What are *regime rays*? permuted detail of "*proper rays*" in *pathtime*-explicit Green's equation variant forms !

#### References

- Aki K., Richards P.G., 1980, Quantitative Seismology Theory and Methods, Freeman and Co. New York.
- Auld, B.A., 1973, Acoustic Fields and Waves in Solids, vol I, John Wiley and Sons, New York.
- Cerveny, V. 2001, Seismic ray theory. Cambridge University Press, Cambridge, U.K.
- Chapman C. H., Pratt R.G., 1992, Travelttime tomography in anisotropic media-I.Theory, Geophys. J. Int. **109** 1-19.
- Gassmann, F., 1964, Introduction to Seismic Travel Time Methods in Anisotropic Media, Pure and Appl Geophysics, **58**, 63-112.
- Helbig, K., 1994, Foundations of Anisotropy for Exploration Seismics, Pergamon
- Helbig, K., Szaraniec, E. 2001, The historical roots of seismic anisotropy, pp 13-20 in Anisotropy2000: Fractures, Converted Waves, and Case Studies: Proceedings of the Ninth International Workshop on Seismic Anisotropy (9IWSA), SEG 2001.
- Love A.E.H., 1927 ed., A Treatise on the Mathematical Theory of Elasticity ,1927 4th ed Cambridge U. Press (1944 Dover, ch. 8)
- MacBeth C. and Lynn H.B. (ed), 2000, Applied Seismic Anisotropy: Theory, ..., Geophys. Reprints #20, Soc. Expl. Geophys.
- Mensch, T., Rasolofosaon, P., 1997, Elastic-wave velocities in ... of Thomsen's epsilon, delta, gamma: Geophys. J. Int. **128**, 43-64.
- Rudzki, M.P., 1911, Parametrische Darstellung der elastischen Wellen in anisotropen Medien, Bull. Acad. Cracovie 503-536
- Rudzki, M.P., 1913, On Application of Fermat's Principle to anisotropic media (Slawinski, M.A., 2001, translation from French), pp 13-20 in Anisotropy2000: Fractures, Converted Waves, and Case Studies: ...[same as above for Helbig, K., Szaraniec, E.]
- Slawinski, M.A., 2001, translator from French with footnotes: M.P.Rudzki (1913 paper), pp 13-20 in Anisotropy2000: (above)
- Thomsen, L., 2002, Understanding Seismic Anisotropy in Exploration and Exploitation, SEG/ EAGE Short Course Series No.5.
- Vetter, W.J., 1993, The Fermat path for ray and wavefront kinematics ..., **63rd** Ann. Mtg Soc. Expl. Geophys., Abstr. 953-956.
- Vetter, W.J., 1999, Kinematics in 3D anisotropic medium: Snell's law ... , **69th** Ann. Mtg Soc. Expl. Geophys., Abstr. 1887-1891.
- Vetter, W.J., 2004, Modeling kinematics for heterogeneous and anisotropic media, Can. Soc. Expl. Geophys. CD Techn. Abstr.

**Appendix: Analytical Detail and Principal Tasks Flow**

$$t v_N (n_x, n_y, n_z ; c_{ij}, \rho) = \mathbf{r} \cdot \mathbf{n} = (x n_x + y n_y + z n_z) \quad \text{Green's front progression equation} \quad (1a)$$

$$t v_N (\theta, \phi ; c_{ij}, \rho) = x \cos\phi \sin\theta + y \sin\phi \sin\theta + z \cos\theta \quad (1b)$$

$$\frac{\partial t}{\partial \theta} = 0 = -\frac{1}{v_N^2} \frac{\partial v_N}{\partial \theta} (x \sin\theta \cos\phi + y \sin\theta \sin\phi + z \cos\theta) + \frac{1}{v_N} (-x \cos\phi \cos\theta + y \sin\phi \cos\theta - z \sin\theta) \quad (2)$$

$$(2) \frac{\partial t}{\partial \phi} = 0 = -\frac{1}{v_N^2} \frac{\partial v_N}{\partial \phi} (x \sin\theta \cos\phi + y \sin\theta \sin\phi + z \cos\theta) + \frac{1}{v_N} (-x \sin\phi \sin\theta + y \cos\phi \sin\theta) \quad (3)$$

$$\begin{pmatrix} v_N \\ \frac{\partial v_N}{\partial \theta} \\ \frac{1}{\sin\theta} \frac{\partial v_N}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\ \cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \frac{x}{t} \\ \frac{y}{t} \\ \frac{z}{t} \end{pmatrix} \quad (4) \quad \begin{pmatrix} \frac{x}{t} \\ \frac{y}{t} \\ \frac{z}{t} \end{pmatrix} = \begin{pmatrix} \sin\theta \cos\phi & \cos\theta \cos\phi & -\sin\phi \\ \sin\theta \sin\phi & \cos\theta \sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} v_N \\ \frac{\partial v_N}{\partial \theta} \\ \frac{1}{\sin\theta} \frac{\partial v_N}{\partial \phi} \end{pmatrix} \quad (5)$$

$$\begin{pmatrix} \frac{\partial v_N}{\partial \theta} \\ \frac{1}{\sin\theta} \frac{\partial v_N}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial v_N}{\partial n_x} \\ \frac{\partial v_N}{\partial n_y} \\ \frac{\partial v_N}{\partial n_z} \end{pmatrix} \quad (6) \quad \begin{pmatrix} \frac{x}{t} \\ \frac{y}{t} \\ \frac{z}{t} \end{pmatrix} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} * v_N + \begin{pmatrix} 1-n_x^2 & -n_x n_y & -n_x n_z \\ -n_x n_y & 1-n_y^2 & -n_y n_z \\ -n_x n_z & -n_y n_z & 1-n_z^2 \end{pmatrix} \begin{pmatrix} \frac{\partial v_N}{\partial n_x} \\ \frac{\partial v_N}{\partial n_y} \\ \frac{\partial v_N}{\partial n_z} \end{pmatrix} \quad (7)$$

$$\left(\frac{x}{t}\right)^2 + \left(\frac{y}{t}\right)^2 + \left(\frac{z}{t}\right)^2 = v_E^2 = v_N^2 + \left(\frac{\partial v_N}{\partial \theta}\right)^2 + \left(\frac{1}{\sin\theta} \frac{\partial v_N}{\partial \phi}\right)^2 = v_N^2 + \left(\sqrt{\left(\frac{\partial v_N}{\partial \theta}\right)^2 + \left(\frac{1}{\sin\theta} \frac{\partial v_N}{\partial \phi}\right)^2}\right)^2 \quad (8)$$

$$\mathbf{r}_{AC} = t * \mathbf{v}_E = t * (v_N \bar{\mathbf{I}}_N + \frac{\partial v_N}{\partial \theta} \bar{\mathbf{I}}_\theta + \frac{1}{\sin\theta} \frac{\partial v_N}{\partial \phi} \bar{\mathbf{I}}_\phi) \quad (9)$$

**TASK-1:** Given  $(a_{ij}, \rho)$  & assigned front-normal direction  $\mathbf{n} = [n_x, n_y, n_z] \Rightarrow \Gamma([n_x, n_y, n_z]; a_{ij}, \rho)$  Christoffel matrix or spherical coord representation  $(\theta, \phi) \Rightarrow \Gamma(\theta, \phi; a_{ij}, \rho)$   
 $\Rightarrow \det(\Gamma - M_m * \text{unit-matrix}) = 0$  or  $M^3 - M^2 \text{tr}(\Gamma) + M * \text{tr}(\text{adj}(\Gamma)) - \det(\Gamma) = 0$  (cubic for linearization)  $\Rightarrow M_m$  moduli &  $(\Gamma - M_m * \text{unit-matrix}) \mathbf{p}_m = 0 \Rightarrow \mathbf{p}_m$  polarization-vectors;  $M_m$  and  $\mathbf{p}_m$  are mode-specific  
 $\Rightarrow M_m$  moduli:  $M_m = \rho v_{Nm}^2$  &  $v_{Nm} = (M_m / \rho)^{1/2}$  with  $m = (qP, qSV, qSH \text{ modes})$  ;  
 $\Rightarrow v_N (n_x, n_y, n_z; a_{ij}, \rho)$  or  $v_N (\theta, \phi; a_{ij}, \rho)$  &  $\partial M / \partial \mathbf{n}$  &  $\partial v_N / \partial \mathbf{n} = (M^* \rho)^{-1/2}$  front-normal velocity & derivatives  
 $\Rightarrow \mathbf{v}_E = (\text{eqn 5, 7})$  energy transport velocity; then for given  $t = t_{\text{FRONT}} \Rightarrow \mathbf{r}_E = t * \mathbf{v}_E$  &  $\mathbf{v}_N = v_N * \mathbf{n}$  &  $\mathbf{r}_N = t * \mathbf{v}_N$

**TASK-2: regime rays:** mode-specif. vectored  $\mathbf{v}_E$  &  $\mathbf{v}_N$  or  $\mathbf{r}_E$  &  $\mathbf{r}_N$  are linked/constrained by eqns { 5, 7, 8, 9} after task-1 pairing; inequality conditions (eqns 11-16) identify the applicable regime code and regime ray expressions;  
 CODE: (1st) {N} segment in 3D along-front-normal, (2nd) {X or Y or Z} in-coordinate-frame plane, (3rd) {x or y or z} along coord-axes directns.  $\Rightarrow \{l_1, l_2, l_3\}$  &  $l_{\text{PATH}} = l_1 + l_2 + l_3$ ;  $\{t_1, t_2, t_3\}$  &  $t_{\text{PATH}} = t_1 + t_2 + t_3$ ;  $\{v_1 = v_N = l_1/t_1, v_2 = l_2/t_2, v_3 = l_3/t_3\}$  &  $\{v_{\text{APNT}} = |\mathbf{v}_E|; v_{\text{TimeAve}} = l_{\text{PATH}} / t_{\text{PATH}}; v_{\text{RMS}} = (l_1^2 v_1 + l_2^2 v_2 + l_3^2 v_3) / (l_1 + l_2 + l_3); v_{\text{PathMean}} = v_{\text{RMS}}^2 / v_{\text{TA}} = t_{\text{PATH}} * v_{\text{RMS}}^2 / l_{\text{PATH}}\}$  ; these are significant velos of along-paths-distributions, also info for along-proper rays quantified heterogeneity from anisotropy (here not elaborated). Note  $v_N < (|\mathbf{v}_E| = v_{\text{APNT}}) < v_{\text{TA}} < v_{\text{RMS}} < v_{\text{PathMean}}$  and  $|\mathbf{r}_N| < |\mathbf{r}_E| < l_{\text{PATH}}$ . Regimes {NZ, NX, NY, Nx, Ny, Nz, Xy, Xz, Yz, Yx, Zx, Zy} included, but best done separately (not shown); {N, X, Y, Z, x, y, z} have  $\mathbf{r}_N = \mathbf{r}_E$ .

**REGIME RAYS: 3D Snell's law representation paths (direction- and velocity- consistent infinitessimals aggregated)** (10)  
unit vectors:  $\{\mathbf{1}_x, \mathbf{1}_y, \mathbf{1}_z\}$  observation coordinate frame aligned with anisotropy structure-natural (or crystal-natural) frame;  
 $\mathbf{1}_N = \mathbf{n} = (\sin\theta \cos\phi) \mathbf{1}_x + (\sin\theta \sin\phi) \mathbf{1}_y + (\cos\theta) \mathbf{1}_z$   $v_N = v_N$  (in 3D general directions)  
 $\mathbf{1}_z = (\cos\phi) \mathbf{1}_x + (\sin\phi) \mathbf{1}_y$   $v_z = v_N / \sin\theta$  (in planes  $z = \text{constant}$ )  
 $\mathbf{1}_x = ((\tan\theta \sin\phi) \mathbf{1}_y + \mathbf{1}_z) / (1 + \tan^2\theta \sin^2\phi)^{1/2}$   $v_x = v_N / \cos\theta (1 + \tan^2\theta \sin^2\phi)^{1/2}$  (in planes  $x = \text{constant}$ )  
 $\mathbf{1}_y = ((\tan\theta \cos\phi) \mathbf{1}_x + \mathbf{1}_z) / (1 + \tan^2\theta \cos^2\phi)^{1/2}$   $v_y = v_N / \cos\theta (1 + \tan^2\theta \cos^2\phi)^{1/2}$  (in planes  $y = \text{constant}$ )  
 $\mathbf{1}_x$   $v_x = v_N / \sin\theta \cos\phi$  (along  $\mathbf{1}_x$  directions)  
 $\mathbf{1}_y$   $v_y = v_N / \sin\theta \sin\phi$  (along  $\mathbf{1}_y$  directions)  
 $\mathbf{1}_z$   $v_z = v_N / \cos\theta$  (along  $\mathbf{1}_z$  directions)

$$(1) \text{ NZX : } (\underline{x_N < x_E}, y_N > y_E, z_N > z_E); \quad y_E > z_E \sin\theta \tan\theta; \quad x_E > y_E / \tan\phi \quad (11)$$

$$\begin{aligned} r_{OE} = l_{NZx} &= (z_E / \cos\theta) \mathbf{1}_N & + & (y_E / \sin\phi - z_E \tan\theta) \mathbf{1}_Z & + & (x_E - y_E / \tan\phi) \mathbf{1}_x \\ l_{OE} = l_{NZx} &= (z_E / \cos\theta) & + & (y_E / \sin\phi - z_E \tan\theta) & + & (x_E - y_E / \tan\phi) \\ t_{OE} = t_{NZx} &= (z_E / \cos\theta) / v_N & + & (y_E / \sin\phi - z_E \tan\theta) / v_Z & + & (x_E - y_E / \tan\phi) / v_x \end{aligned}$$

$$(2) \text{ NZy : } (x_N > x_E, \underline{y_N < y_E}, z_N > z_E); \quad x_E > z_E \tan\theta \cos\phi; \quad y_E > x_E \tan\phi \quad (12)$$

$$t_{OE} = t_{NZy} = (z_E / \cos\theta) / v_N + (x_E / \cos\phi - z_E \tan\theta) / v_Z + (y_E - x_E \tan\phi) / v_y$$

$$(3) \text{ NXy : } (x_N > x_E, \underline{y_N < y_E}, z_N > z_E); \quad z_E \tan\theta \cos\phi > x_E; \quad y_E > z_E \tan\theta \sin\phi \quad (13)$$

$$t_{OE} = t_{NXy} = (x_E / \sin\theta \cos\phi) / v_N + (z_E - x_E / \tan\theta \cos\phi) (1 + \tan^2\theta \sin^2\phi)^{1/2} / v_x + (y_E - z_E \tan\theta \sin\phi) / v_y$$

$$(4) \text{ NXz : } (x_N > x_E, y_N > y_E, \underline{z_N < z_E}); \quad y_E > x_E \tan\phi; \quad z_E \tan\theta \sin\phi > y_E \quad (14)$$

$$t_{OE} = t_{NXz} = (x_E / \sin\theta \cos\phi) / v_N + (y_E / \tan\theta \sin\phi - x_E / \tan\theta \cos\phi) (1 + \tan^2\theta \sin^2\phi)^{1/2} / v_x + (z_E - y_E / \tan\theta \sin\phi) / v_z$$

$$(5) \text{ NYx : } (\underline{x_N < x_E}, y_N > y_E, z_N > z_E); \quad z_E \tan\theta \sin\phi > y_E; \quad x_E > z_E \tan\theta \cos\phi \quad (15)$$

$$t_{OE} = t_{NYx} = (y_E / \sin\theta \sin\phi) / v_N + (z_E - y_E / \tan\theta \sin\phi) (1 + \tan^2\theta \cos^2\phi)^{1/2} / v_y + (x_E - z_E \tan\theta \cos\phi) / v_x$$

$$(6) \text{ NYz : } (x_N > x_E, y_N > y_E, \underline{z_N < z_E}); \quad x_E \tan\phi > y_E; \quad z_E \tan\theta \cos\phi > x_E \quad (16)$$

$$t_{OE} = t_{NYz} = (y_E / \sin\theta \sin\phi) / v_N + (x_E / \cos\phi \tan\theta - y_E / \sin\phi \tan\theta) (1 + \tan^2\theta \cos^2\phi)^{1/2} / v_y + (z_E - x_E / \tan\theta \cos\phi) / v_z$$