On Choosing Differential Operators

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Summary

Differential operators are used in many seismic data processes such as triangle filters to reduce aliasing, finite difference solutions to the wave equation, or wavelet correction when modelling with diffractions or migrating with Kirchhoff algorithms. Short operators (two to nine samples) may be quite accurate when the data are restricted to locally low order polynomials, or may be inaccurate in other applications.

The first derivative, the second derivative, and the square-root derivative or rho filter are discussed with the main emphasis placed on the first derivative, using a polynomial and spectrally derived approximations. The accuracy of the operators is compared using spectral amplitudes in the frequency domain.

Introduction

I use an array of data f_n, that I call a trace, that was obtained from a continuous function $f(t)$ defined in time. Seismic data is typically sampled in time with a maximum frequency content well below the Nyquist frequency. The data is then locally smoothed and can be fitted locally with a low order polynomial. (Unfortunately, the spatially sample data is typically under-sampled and may be aliased.)

Comparisons of operators are made in the frequency domain and displayed with the positive frequencies in the first half of the image, and the negative frequencies in the second half, with the Nyquist frequency in the center. I typically use N_{fft} = 512 so that samples from 1 to 256 will roughly correspond to frequencies from 0 to 250 Hz.

Approximations to the First derivative using polynomial approximations

Assume that a small section of the trace can be represented by a polynomial, and that we can define a shifted value of the data using the Taylor series, i.e.,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots \quad . \tag{1}$$

We can assume the second and higher derivative are zero and the above equation can be rearranged to give the forward approximation to the derivative

$$f_f'(x) = \frac{-f(x) + f(x+h)}{h} \, . \tag{2}$$

Changing the sign of h in the Taylor series we have

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \ldots \quad , \tag{3}$$

and assuming again that the second and higher derivatives are zero we get the backward difference

$$f_b^{!}\left(x\right) = \frac{-f\left(x-h\right)+f\left(x\right)}{h}. \tag{4}$$

Subtracting equations (3) from (1) we get

$$-f\left(x-h\right)+f\left(x+h\right) = 2hf'\left(x\right)+2\frac{h^3}{3!}f'''\left(x\right)+..., \tag{5}$$

and ignoring the higher derivative we get a third definition for the derivative, the central difference equation defined over three points, however the central value is zero, i.e.,

$$f_c^{!}\left(x\right) = \frac{-f\left(x-h\right)-f\left(x+h\right)}{2h}. \tag{6}$$

These three finite difference approximations to the first derivative are different in the phase and amplitude response. The first two approximations of equations (2) and (4) have a half sample shift in opposite directions that produces linear phase shifts. The third approximation in equation (6) produces the correct phase as the data is centered about the location of the derivative, but has a lower bandwidth.

Now assume the second order polynomial below in equation (7) that contains a real value for the second derivative, i.e. 2*a*, along with values defined at (*x*+*h*) and (*x*+2*h*) i.e.,.

$$f\left(x\right) = a\left(x\right)^2+b\left(x\right)+c, \tag{7}$$

$$f\left(x+h\right) = a\left(x+h\right)^2+b\left(x+h\right)+c, \tag{8}$$

$$f\left(x+2h\right) = a\left(x+2h\right)^2+b\left(x+2h\right)+c. \tag{9}$$

We can solve these equations to get the constants *a*, *b*, and *c* in terms of the sample locations, however we only need 2*a* as it is the only remaining value when we take the second derivative of (7). We can now substitute this value into the Taylor series of equation (1) that includes the second derivative to get a new estimate of the first derivative

$$f'\left(x\right)_{\text{Left}} = \frac{-3f\left(x\right)+4f\left(x+h\right)-f\left(x-h\right)}{2h}. \tag{10}$$

Notice that the solution is at the location of the first sample (*x*) or the left side of the samples, and can be used to get the derivative at the first sample of a trace. We could also have used the above process using function values at (*x*-*h*), (*x*), and (*x*+*h*) to get a central value, or (*x*-2*h*), (*x*-*h*), (*x*) to get a right sided estimate. The central value will return a result identical to that of equation (6) and can be used for the central samples of a trace, while the right sided solution will be valid at the end of the trace.

We can continue to add more samples in similar configurations to get higher order approximations for the first derivative. The following are an example of a left sided fifth order and a central seventh order approximation, i.e.,

$$f'_{5left}\left(x\right) = \frac{-25f\left(x\right)+48f\left(x+h\right)-36f\left(x+2h\right)+16f\left(x+3h\right)-3f\left(x+4h\right)}{12h}, \text{ and} \tag{11}$$

$$f'_{7central}\left(x\right) = \frac{-f\left(x-3h\right)+9f\left(x-2h\right)-45f\left(x-h\right)+45f\left(x+h\right)-9f\left(x+2h\right)+f\left(x+3h\right)}{60h}. \tag{12}$$

Tables of these finite difference approximations are available on the internet as indicated by Ref1. The above derivation was included to illustrate theoretical estimates of the first derivative. However there are a number of alternate methods such as spectral approximations, using the Remez algorithm (available in MATLAB), or a recursive type filter, and are discussed in more detail in Bancroft and Geiger (2003) and Bancroft (2008a, b, and c).

Approximations to the First derivative using spectral approximations

The spectral method defines derivative in the frequency domain where the data was tapered in the area of the Nyquist frequency, inverse transformed back to the time domain, windowed with a cosine or raised cosine window to produce an operator with only a few samples. The size of the tapering and the window are altered to obtain an optimum result, and are compared with the results obtained from the polynomial approximations below.

Approximations for the second derivative can be obtained in a similar manner using polynomial and frequency domain approximations. The square-root derivative or rho filter is a much more difficult to estimate and requires more samples to obtain a reasonable approximation.

Examples

A spectral definition of the derivative is shown in Figure 1a that shows the only imaginary component in blue as defined by $j\omega$. Note the extreme change at the Nyquist frequency in the center. A taper starting at 50% of the Nyquist frequency, shown in green (scaled by 10) is applied to give the designed spectral shape in red. The theoretical inverse transform is shown in Figure 1b, and the tapered response in (c). The tapered response appears to require much fewer samples, and may be approximated with say 7 samples.

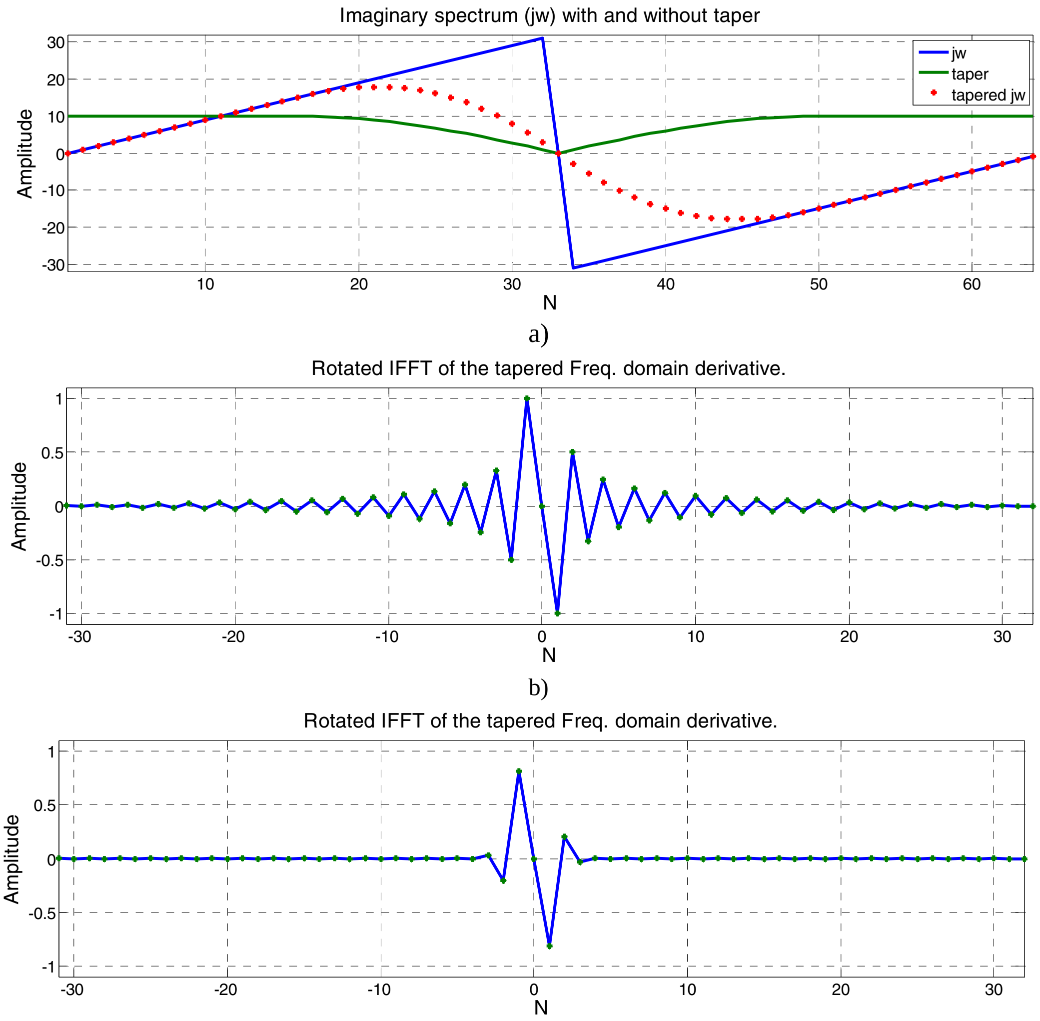

Figure 1: Spectral definition of the derivative in a) showing the theoretical and tapered imaginary spectrums, b) the theoretical time domain response, and c) the tapered time response, both with time zero at the center.

We can continue to reduce the number of samples by applying a cosine window about the central samples as illustrated in Figure 2 that used N_{fft} 1024 complex samples and a taper starting at 80% of the Nyquist frequency. Figure 3 shows the amplitude spectra for 3, 5, 7, and 9 point spectrally derived operators while Figure 4 shows the amplitude spectra for 3, 5, and 7 point polynomial operators.

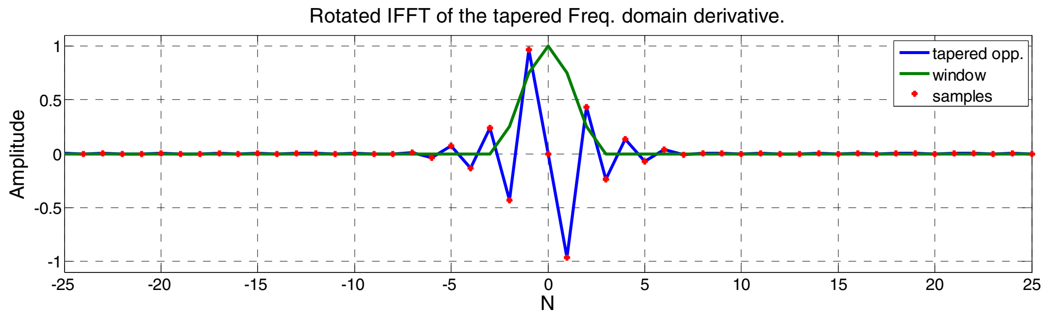

Figure 2 Inverse Fourier transform of the derivative using the tapered frequency, N_{fft} = 1024, taper starting at 80% of the Nyquist frequency, and a 5 point window.

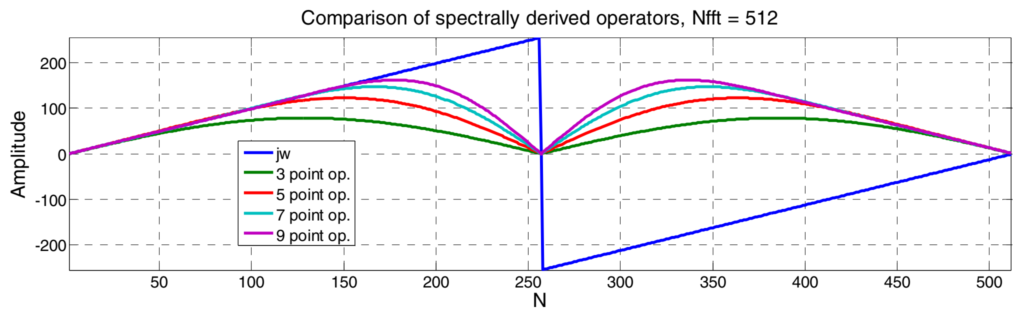

Figure 3 Comparison of the amplitude spectra for spectrally derived operators for operator sizes 3, 5, 7, and 9 points

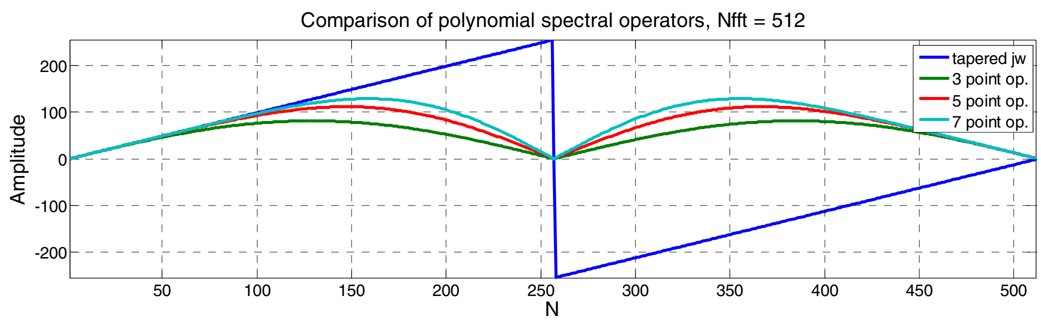

Figure 4 Comparison of the amplitude spectra for polynomial derived operators for operator sizes 3, 5, and 7 points.

Conclusions and Discussion

Short finite difference approximations to the first derivative were demonstrated using the Taylor series and polynomial approximations, and compared with spectrally defined operators. Visual inspection of the amplitude spectra show that the spectrally defined operators have a greater bandwidth, however the error in the pass band is greater though it was not shown. The same principles apply to the design of second derivative and square-root derivative operators.

Acknowledgements

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References

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