Dynamic raytracing in inhomogeneous anisotropic media

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Summary

I present a new differential equation system for dynamic raytracing in inhomogeneous anisotropic media. This new system is computationally much more efficient than the traditional elastic-parameter based system as the right-hand-side functions of this new system are simple. Evaluation of these functions involves only differentiation of phase velocities. Evaluation of the right-hand-side functions of the traditional system, on the other hand, requires solving an eigenvalue problem and differentiating the resulting eigenvectors at each ray step, both being time-consuming processes. The new system is also simpler and more efficient than a previously proposed phase-velocity based dynamic raytracing system since the former is formulated in orthogonal coordinates while the latter in nonorthogonal ones.

Introduction

Dynamic raytracing is an essential building block for seismic modeling and imaging with ray methods. It is needed, for example, for calculating ray amplitudes and for constructing beam solutions to wave equations. Dynamic raytracing in inhomogeneous anisotropic media has traditionally been formulated in terms of elastic parameters (e.g., Červený, 1972). Such a formulation results in a system with complicated right-hand-side functions and is computationally cumbersome (Červený, 1989). To overcome this difficulty, Zhu et al. (2005) developed a phase-velocity based dynamic ray tracing system which is much simpler and computationally more efficient than the elastic-parameter based one. This system, however, suffers a drawback in that it is formulated in nonorthogonal coordinates which complicate the computation. The purpose of this study is to further simplify the phase-velocity based dynamic raytracing system by formulating it in orthogonal coordinates and by employing a phase-velocity based Hamiltonian. The latter also enables us to derive, for the first time, an explicit formula for calculating group velocity components from phase velocities for arbitrary anisotropic media, which greatly facilitates the implementation of the phase-velocity based raytracing systems in an anisotropic medium.

Elastic-parameter based dynamic raytracing system

The elastic-parameter based dynamic raytracing system in anisotropic media has been discussed in detail by Červený (1972) and many others. I summarize here only the results used in this study. Denote the traveltime along a given ray by $\tau(x_i)$ and the corresponding slowness vector by $p_i^{(x)} = \partial\tau/\partial x_i$, where x_i represents the global Cartesian coordinates. For an inhomogeneous anisotropic medium described by the density normalized elastic parameter a_{ijkl} , the ray approximation to the solution of the elastic wave equation leads to the Christoffel equation:

$$\left(\Gamma_{jk} - \delta_{jk}\right) g_k = 0, \qquad (1)$$

where $\Gamma_{jk} = a_{ijkl} p_i^{(x)} p_l^{(x)}$ is the Christoffel matrix. Equation (1) describes an eigenvalue problem for matrix Γ_{jk} , and g_k is its normalized eigenvector. The three eigenvalues of this equation take the form $G(x_i, p_i^{(x)}) = 1$, which is an eikonal equation for a given wave type. This eikonal equation can be solved using the Hamiltonian

$$H(x_{i,} p_i^{(x)}) = \left(G(x_i, p_i^{(x)}) - 1\right)/2 \,, \qquad (2)$$

yielding the kinematic raytracing system:

$$\frac{dx_i}{d\tau} = \frac{\partial H}{\partial p_i^{(x)}} = \frac{1}{2}\frac{\partial G}{\partial p_i^{(x)}} = a_{ijkl} p_l^{(x)} g_j g_k\,, \quad \frac{dp_i^{(x)}}{d\tau} = -\frac{\partial H}{\partial x_i} = -\frac{1}{2}\frac{\partial G}{\partial x_i} = -\frac{1}{2}\frac{\partial a_{mjkl}}{\partial x_i} p_m^{(x)} p_l^{(x)} g_j g_k\,. \tag{3}$$

The indices in (3) take values 1, 2, 3, and a repeated index indicates a summation over these values, resulting in a total number of 144 independent terms in the right-hand-side summations (Červený, 1989).

Consider the ray coordinates $(\gamma_1, \gamma_2, \tau)$ where γ_1 and γ_2 are the ray parameters that specify a ray, and τ is the traveltime along the ray. The elastic-parameter based dynamic raytracing system in the global Cartesian coordinates is obtained by differentiating the equations in (3) with respect to the ray parameters. This leads to a system of six linear first-order differential equations:

$$\frac{dQ_i^{(x)}}{d\tau} = \frac{\partial^2 H}{\partial p_i^{(x)} \partial x_j} Q_j^{(x)} + \frac{\partial^2 H}{\partial p_i^{(x)} \partial p_j^{(x)}} P_j^{(x)}, \quad \frac{dP_i^{(x)}}{d\tau} = -\frac{\partial^2 H}{\partial x_i \partial x_j} Q_j^{(x)} - \frac{\partial^2 H}{\partial x_i \partial p_j^{(x)}} P_j^{(x)}, \tag{4}$$

where $Q_i^{(x)} = \partial x_i / \partial \gamma$, $P_i^{(x)} = \partial p_i / \partial \gamma$, and γ represents γ_1 or γ_2. The second derivatives of the Hamiltonian H in (4) can be obtained by differentiating the right-hand-side functions of (3). Since these functions are multiplications of elastic parameters and their derivatives with eigenvectors, the explicit expressions for these second derivatives are long and will not be given here. For the explicit form of the right-hand-side functions of (4), the reader is referred to Červený (1972).

Phase-velocity based dynamic raytracing system

To derive the phase-velocity based dynamic ray tracing system, consider a Christoffel equation with a slightly different form than equation (1). This is done by defining

$$p_i^{(x)} = n_i / v\,, \tag{5}$$

where phase normal n_i is the unit vector along the slowness vector and $v = v(x_i, n_i)$ is the phase velocity. Substituting (5) into (1) gives

$$\left(\Gamma_{jk} - v^2 \delta_{jk}\right) g_k = 0, \tag{6}$$

where the Christoffel matrix is now given by $\Gamma_{jk} = a_{ijkl} n_i n_l$ .Solution of (6) yields three eigenvalues $v = v(x_i, n_i)$ which correspond to the phase velocities of the P-wave and two S waves along the direction **n** (e.g., Tsvankin, 2001). Combing equation (5) and $p_i^{(x)} = \partial\tau / \partial x_i$ gives the eikonal equation $(\nabla \tau)^2 = 1 / v^2(x_i, n_i)$ which has the same form as its counterpart in isotropic media except that the phase velocity v is now a function of both position and direction instead of position alone as in isotropic media. It can be solved similarly using a phase-velocity based Hamiltonian

$$K(x_i, p_i^{(x)}) = \left(v^2(x_i, n_i) p_k^{(x)} p_k^{(x)} - 1\right) / 2\,. \tag{7}$$

The corresponding kinematic raytracing system is given by

$$\frac{dx_i}{d\tau} = \frac{\partial K}{\partial p_i^{(x)}} = v^2 p_i^{(x)} + \frac{1}{v}\frac{\partial v}{\partial p_i^{(x)}}\,, \tag{8a}$$

$$\frac{dp_i^{(x)}}{d\tau} = -\frac{\partial K}{\partial x_i} = -\frac{1}{v}\frac{\partial v}{\partial x_i}\,. \tag{8b}$$

Specify the phase normal as $\mathbf{n} = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$, where ϕ and θ are the azimuthal and polar angles, respectively. It can then be shown that equation (8a) can be written as

$$\begin{pmatrix} dx_1/d\tau \\ dx_2/d\tau \\ dx_3/d\tau \end{pmatrix} = \begin{pmatrix} \cos\phi\sin\theta & \cos\phi\cos\theta & -\sin\phi/\sin\theta \\ \sin\phi\sin\theta & \sin\phi\cos\theta & \cos\phi/\sin\theta \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} v \\ \partial v/\partial\theta \\ \partial v/\partial\phi \end{pmatrix} \tag{9}$$

Since, by definition, the group velocity component $V_i^{(x)} = dx_i / d\tau$, equation (9) can also be used to determine group velocity from phase velocity for other applications. Ray equation (8a) was also derived by Slawinski (2003) with a Hamiltonian similar to that in (7), and used to calculate group velocities for 2D anisotropic media. Equation (9), however, provides for the first time an explicit formula for calculating group velocity components from phase velocity in an arbitrary 3D inhomogeneous anisotropic medium.

Differentiation of equation (8) with respect to ray parameters yields the phase-velocity based dynamic raytracing system in the global Cartesian coordinates:

$$dQ_i^{(x)} / d\tau = A_{ij}^{(x)} Q_j^{(x)} + B_{ij}^{(x)} P_j^{(x)}, \quad dP_i^{(x)} / d\tau = -C_{ij}^{(x)} Q_j^{(x)} - D_{ij}^{(x)} P_j^{(x)}, \ \ (i, j = 1,2,3)\,, \tag{10}$$

where

$$A_{ij}^{(x)} = 2v \frac{\partial v}{\partial x_j} p_i^{(x)} + \frac{1}{v^2} \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial p_i^{(x)}} + \frac{1}{v} \frac{\partial^2 v}{\partial x_j \partial p_i^{(x)}}$$

$$B_{ij}^{(x)} = 2v \left(\frac{\partial v}{\partial p_j^{(x)}} p_i^{(x)} + \frac{\partial v}{\partial p_i^{(x)}} p_j^{(x)} \right) + v^2 \delta_{ij} + \frac{1}{v^2} \frac{\partial v}{\partial p_j^{(x)}} \frac{\partial v}{\partial p_i^{(x)}} + \frac{1}{v} \frac{\partial^2 v}{\partial p_i^{(x)} \partial p_j^{(x)}}$$

$$C_{ij}^{(x)} = \frac{1}{v^2} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} + \frac{1}{v} \frac{\partial^2 v}{\partial x_i \partial x_j}, \text{ and } D_{ij}^{(x)} = \frac{1}{v^2} \frac{\partial v}{\partial p_j^{(x)}} \frac{\partial v}{\partial x_i} + \frac{1}{v} \frac{\partial^2 v}{\partial x_i \partial p_j^{(x)}} + 2 \frac{\partial v}{\partial x_i} p_j^{(x)}.$$

For many applications such as beam calculation, it is convenient to transform the dynamic raytracing system in (10) from the global Cartesian coordinates to local coordinates centered along a reference ray. Such local coordinates also reduce the number of differential equations in (10) from six to four. Two different local coordinate systems have been proposed: the ray-centered and the wavefront orthonormal coordinate systems (e.g., Hanyga, 1986; Bakker, 1996; Červený, 2001). The latter is used in this study since the former is nonorthogonal, and construction of transformation matrices between this and the Cartesian coordinate systems requires calculation of both contravariant and covariant basis vectors. The wavefront system, on the other hand, is orthonormal and requires the calculation of only one set of basis vectors. Denote the wavefront orthonormal coordinates by $\mathbf{y} = (y_1, y_2, y_3)$ and the corresponding slowness vector by $p_i^{(y)} = \partial\tau / \partial y_i$. The origin of the coordinates moves along the reference ray and is located at the intersection point between the wavefront and the ray. The basis vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are chosen in such a way that $\mathbf{e}_3 = \mathbf{n}$ is perpendicular to the wavefront, and $\mathbf{e}_1$ and $\mathbf{e}_2$ are tangent to the wavefront. Basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are determined along the ray by solving the differential equation

$$d\mathbf{e}_I / d\tau = v(\mathbf{e}_I \bullet \nabla v)\mathbf{p}^{(x)}, \ \ (I = 1,2). \tag{11}$$

Using the transformation matrix formed by the components of the resulting basis vectors, one can transform system (10) from the global Cartesian coordinates to the wavefront orthonormal coordinates. The resulting phase-velocity based dynamic raytracing system in the wavefront orthonormal coordinates consists of four linear first-order differential equations:

$$dQ_M^{(y)} / d\tau = A_{MN}^{(y)} Q_N^{(y)} + B_{MN}^{(y)} P_N^{(y)}, \quad dP_M^{(y)} / d\tau = -C_{MN}^{(y)} Q_N^{(y)} - D_{MN}^{(y)} P_N^{(y)}, \ \ (M, N = 1,2)\,, \tag{12}$$

where $Q_M^{(y)} = \partial y_M / \partial\gamma$ and $P_M^{(y)} = \partial p_M^{(y)} / \partial\gamma$, and the coefficients are given by

$$A_{MN}^{(y)} = \frac{1}{v^2} \frac{\partial v}{\partial p_M^{(y)}} \frac{\partial v}{\partial y_N} + \frac{1}{v} \frac{\partial^2 v}{\partial p_M^{(y)} \partial y_N} - \frac{1}{v} \frac{\partial v}{\partial y_N} V_M^{(y)}$$

$$B_{MN}^{(y)} = v^2\delta_{MN} + \frac{1}{v^2}\frac{\partial v}{\partial p_M^{(y)}}\frac{\partial v}{\partial p_N^{(y)}} + \frac{1}{v}\frac{\partial^2 v}{\partial p_M^{(y)}\partial p_N^{(y)}} - V_M^{(y)}V_N^{(y)}$$

$$C_{MN}^{(y)} = \frac{1}{v}\frac{\partial^2 v}{\partial y_M \partial y_N}, \text{ and } D_{MN}^{(y)} = \frac{1}{v^2}\frac{\partial v}{\partial y_M}\frac{\partial v}{\partial p_N^{(y)}} + \frac{1}{v}\frac{\partial^2 v}{\partial y_M \partial p_N^{(x)}} - \frac{1}{v}\frac{\partial v}{\partial y_M}V_N^{(y)},$$

where $V_M^{(y)}$ and $V_N^{(y)}$ represent the group velocity components in y_1 or y_2 directions. In an isotropic medium $V_M^{(y)} = V_N^{(y)} = 0$, phase velocity is no longer the function of $p_i^{(y)}$, and (12) reduces to the well-known dynamic raytracing system in the isotropic ray-centered coordinates.

Dynamic raytracing system (12) is much simpler and computationally more efficient than system (4) since that elastic-parameter based system involves differentiation of the complicated functions on the right-hand sides of equations (3). These functions are summations of a large number of terms and each term includes the components of eigenvectors. Differentiation of these eigenvectors is complicated and time-consuming. The right-hand sides of system (12), on the other hand, have only a small number of terms and evaluation of these terms requires only simple calculation of phase velocities and their derivatives. A phase-velocity based dynamic ray tracing system was also derived by Zhu et al. (2005) using Hamiltonian (2). Because of the use of this Hamiltonian, however, they were unable to obtain an explicit formula for calculating group velocity components from phase velocity, limiting the application of their method in general anisotropic media. Also, their dynamic raytracing system was formulated in the ray-centered coordinates and is hence less efficient than that in (12).

Conclusions

I have developed a new system for dynamic ray tracing in inhomogeneous anisotropic media. Formulated in terms of phase velocity, this system is much simpler and computationally more efficient than that formulated previously in terms of elastic parameters, since calculation of the right-hand side of the new system now involves only simple evaluation of phase velocities and their derivatives rather than complicated functions for the elastic-parameter based system. Complicated and time-consuming differentiation of eigenvectors is no longer needed. The new system is also simpler than the phase-velocity based dynamic raytracing system proposed previously by Zhu et al. (2005) as it is formulated in the orthogonal wavefront orthonormal coordinates rather than in the nonorthogonal ray-centered coordinates. Moreover, by using a phase-velocity based Hamiltonian, I derived, for the first time, an explicit formula for calculating group velocity components from phase velocity for arbitrary 3D inhomogeneous anisotropic media. This, together with the explicit expressions given in (12) for the right-hand-side functions, greatly facilitates the implementation of the dynamic raytracing system. The newly developed dynamic raytracing system thus provides an efficient and practical tool for calculating ray amplitudes and constructing beam solutions for general inhomogeneous, anisotropic media.

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